Method of Frobenius: Equal Roots to the Indicial Equation

We solve the equation \( x^2 y'' + 3 xy' + (1 - x) y = 0 \) using a power series centered at the regular singular point \( x = 0 \). (You should check that zero is really a regular singular point.)

Let \( y = \sum_{n=0}^{\infty} a_n x^n \). Then, inserting this series into the differential equation results in

\[
(r + 1)^2 a_0 x^r + \sum_{n=1}^{\infty} (a_n (n + r + 1)^2 - a_{n-1}) x^{n+r} = 0
\]

The indicial equation is \((r + 1)^2 = 0\) and we have a repeated zero \( r = -1 \). We also have the recurrence relation

\[
a_n = \frac{a_{n-1}}{(n + r + 1)^2}
\]

valid for \( n \geq 1 \).

For \( r = -1 \), we obtain the recurrence relation

\[
a_n = \frac{a_{n-1}}{n^2}
\]

A little work shows that

\[
a_n = \frac{1}{(n!)^2}
\]

where we have set \( a_0 = 1 \). The first solution to the DE is therefore

\[
y_1 = \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}
\]

To find the second solution, we need to solve the general recurrence relation without using \( r = -1 \). In this case the solution is easy to find:

\[
a_n = \frac{a_0}{(r + 2)^2 (r + 3)^2 \cdots (n + r + 1)^2}
\]

We need to calculate \( \frac{d}{dx} (a_n) \mid_{r=-1} \). To do this, we take the natural log of both sides:

\[
\ln a_n = \ln a_0 - 2 \ln(r + 2) + \ln(r + 3) + \ln(r + 3) + \cdots + \ln(r + n + 1)
\]

Now we differentiate with respect to \( r \):

\[
\frac{a_n'}{a_n} = -2 \left( \frac{1}{r + 2} + \frac{1}{r + 3} + \cdots + \frac{1}{n + r + 1} \right)
\]

We can rewrite this as

\[
a_n' = -2 a_n \left( \frac{1}{r + 2} + \frac{1}{r + 3} + \cdots + \frac{1}{n + r + 1} \right)
\]

We now evaluate the derivative at the zero \( r = -1 \) to obtain
The second solution is therefore

\[
y_2 = \left( \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} \right) \ln(x) - \frac{2}{x} \left( 1 + \sum_{n=1}^{\infty} \frac{H_n x^n}{(n!)^2} \right)
\]

where \( H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \).