

WHY SO MUCH STABILITY? A FUZZY SET THEORY RESPONSE

1. Cycling in Spatial Models

William H. Riker (1980) noted that spatial models predicated on rational choice assumptions predict that political life should be rife with cycling (Arrow 1951, Black 1958, Buchanan and Tullock 1962). He based the observation on theoretical research demonstrating the instability of political outcomes decided by majority decision in two-dimensional space, the most important of which was McKelvey's (1976, 1979) discovery that in the absence of highly restrictive conditions on the location of ideal points (Plott's [1967] radial symmetry) a core does not exist under sincere voting. The core, which henceforth we will refer to as a maximal set under majority rule, comprises a mathematically stable set of outcomes. No alternative outside or within this set is strictly preferred by a majority to any alternative in the set; and majorities are indifferent in a choice between alternatives in the set.¹ In the absence of a majority rule maximal set, the best that can be achieved is the prediction of a top cycle set, a set of alternatives that defeat all other alternatives outside the set, but each of which can be defeated by some alternative in the set itself. As McKelvey (1976) demonstrated, the top cycle set encompasses the entire policy space in the absence of radial symmetry (Plott, 1967, Enelow and Hinich, 1983). Essentially, spatial models predict that any outcome in two-dimensional

space is possible under majority rule and sincere voting.

In response to Riker, Gordon Tullock (1981) observed the dearth of empirical evidence of cycling and asked why there was so much stability. Subsequent research largely has gone in two directions in an effort to find a better fit between the predictions made by spatial models and empirical reality. The first focuses on the uncovered set. Its advocates note that McKelvey's argument rests on the assumption that political actors are myopic, voting sincerely in each successive round without regard for past or future votes. Under an assumption of strategic voting, spatial models predict that an outcome will fall somewhere in a smaller region of the policy space, the uncovered set (McKelvey, 1986, Miller, 1980, Miller, 2007, Shepsle and Weingast, 1984). Unfortunately, the location, size, and shape of the uncovered set are often not intuitive and can be very difficult to determine. Recent software developments, such as CyberSenate (see Miller, 2007), have done much to reduce the complexity of the calculations, but Euclidean preferences and equality of dimensional salience must be assumed. Furthermore, there are several different formal definitions of the uncovered set, which can yield different results if political actors are indifferent over any subset of the alternatives (Bordes, Le Breton, and Salles, 1992, Penn, 2006).

The second direction that research has taken, the *structurally induced equilibrium* approach, focuses on identifying institutional designs that assure stable political outcomes by reducing the set of alternatives in the choice set. Some

scholars have also argued that political actors purposely reduce the choice set (Guerdjikova and Zimper, 2008). While more elegant than the uncovered set, the approach is burdened by the requirement for a substantial number of restrictive assumptions. Among the seminal works in this vein is that of Kenneth Shepsle (1979) who argues that stable legislative outcomes result from committee systems that induce dimension-by-dimension consideration of bills. Laver and Shepsle (1996) later applied the same principle to the cabinet formation process in parliamentary government. Their model predicts a majority preferred cabinet at the intersection of the medians on the salient policy dimensions. Its ability to do so is dependent on several assumptions, to include issue separability, Euclidean preferences, and abdication. The latter is the assumption that the allocation of a government ministry to a party involves the surrender of control over policy under the jurisdiction of that ministry on the part of other parties to a coalition government. The abdication assumption has come under particular criticism from empiricists (see for example, Warwick, 1999).

More recently, a third approach has emerged focusing on the effect of preference indifference on the existence of a majority rule maximal set. Several studies have found that the probability of a majority rule maximal set increases when the strict preference relationship is relaxed to permit indifference over alternatives (Tovey, 1991, Skog, 1994, Gehrlein and Fabrice, 2001, Balke, Guntzer and Siberski, 2006, Ehlers and Barbera, 2007). Much of this research makes use of Tovey's (1991)

concept of an epsilon-core (ϵ -core), a threshold distance in Euclidean space that must be exceeded before players distinguish between alternatives (see Koehler, 2001, Bräuninger, 2007). Unless an alternative lies outside of the region defined by the ϵ -core, a player is indifferent between it and the core's center. Actors are essentially indifferent to alternatives in close proximity.

We contribute a fuzzy set theoretic approach to the emerging literature on the effect of indifference on the existence of a majority rule maximal set. We follow the lead of Clark, Larson, Mordeson, Potter, and Wierman (2008) and adopt discrete fuzzy numbers to represent the preferences of political actors. This not only permits us to model a substantial degree of indifference, but in contrast to the ϵ -core concept we can do more than thicken the line that defines a player's win-circle. Instead, we can model uniform indifference over a fixed and discrete region of the policy space. Furthermore, the approach requires neither burdensome assumptions nor complex calculations. Moreover, it can deal with irregular shaped preferences, and it does so in a less cumbersome manner than other mathematical approaches. If spatial models are to be tested empirically, they will need to be able to accommodate any shape imposed upon them by estimates derived from data sets. As yet, no convention has emerged on how best to estimate preferences, but judging by the efforts to do so using Nominat (see for example Bianco, Jeliazkov, and Sened, 2004) and the Comparative Manifesto Project (see Laver, 2001), preferences are likely to take any number of shapes. The oddity of those

shapes is far likelier in the case of collective institutions.

Previous work has demonstrated that fuzzy spatial models can predict a majority rule maximal set when conventional crisp models would not (Clark, Larson, Mordeson, Potter, and Wierman, 2008). In what follows, we consider how well behaved fuzzy models are. We offer a formal proof that in all but a limited number of cases two-dimensional fuzzy set spatial models result in an empty maximal set *if and only if* the Pareto set contains a union of cycles (Theorem 2.2). Moreover, we demonstrate that these same models result in a nonempty maximal set if the Pareto set does not contain a cycle (Theorem 2.3). We conclude by completely characterizing the elements that constitute the exception for a three-person game (Theorem 3.1) based on the general definition for n players provided in Theorem 2.2. Our theorems rely on a general theory in which a region of interest is mapped to a region with a suitable and natural partial ordering. The partial ordering plays a significant role in the determination of the results. The mapping involved then transfers the results faithfully back to the original region of interest. The formal proof of this transfer appears in the appendix. Since the approach is done in general, the ideas developed can be applied to other types of problems. The results have connections to other areas in mathematics.

2. Fuzzy Spatial Modeling

The fuzzy set approach to spatial models assumes that political actors are ambiguous concerning their policy preferences. Individual actors have preferences over all alternatives, each of which is assigned a value over the interval $[0, 1]$ indicating the degree to which she considers it an ideal policy position. In essence, the policy space is partitioned into a finite number of regions within which a player is indifferent among alternatives.

Formally, each element x of the universe X in a fuzzy set \tilde{F} has a degree of membership, $\tilde{F}(x)$, that is specified by a function \tilde{F} from X into the closed interval $[0, 1]$. The set of ideal policy preferences for the actor are assigned the value 1 (full membership in the set of ideal policies), and those that are “not ideal” to any degree are assigned the value 0 (no membership in the set of ideal policies).

All remaining policies are scored on the degree to which they come close to being ideal. The conventional approach, which results in instability, requires us to assume that actors can make infinitely fine grained distinctions between alternatives. A fuzzy approach invites us to consider *granularity*, the precision with which players can distinguish between groups of alternatives. The lesser the degree to which players can discern between alternatives, the fewer the number of groups into which they categorize alternatives and the greater the degree of indifference. Thus, the fuzzy approach not only models preferences with “a plateau

on top” (Black, 1958); but it deals with flat spaces (indifference) at multiple levels of a player’s preference curve, the number of those levels being determined by the granularity.

Without loss of generality, we employ a relatively coarse granularity that has an intuitive appeal, similar to that of a Likert scale. Those alternatives considered almost ideal by political actors are assigned a fuzzy preference score of .75 (three-fourths membership in the set of ideal policies), those considered “neither ideal nor not ideal” are scored .50 (one-half membership in the set of ideal policies), and those “less ideal than ideal” are scored .25 (one-quarter membership in the set of ideal policies).

We are essentially using discrete fuzzy numbers to model indifference resulting from ambiguity over preferences at different levels of set inclusion, or α -levels. All points at any given α -level (an α -slice) are equally preferred to one another. The subset of elements whose set membership grade is exactly α comprises an **alpha-slice**. An α -slice of a fuzzy set \tilde{F} is the set of all elements that have a membership grade of α , $\tilde{F}_\alpha = \{x | \tilde{F}(x) = \alpha\}$. Moreover, no restriction is artificially placed on the shape of players’ preferences. The Euclidean distance assumption imposes a circular (or minimally an elliptical) shape on players’ preference. The fuzzy approach can even accommodate concave shaped preferences.

Clark, et.al. (2008) use these fuzzy concepts to map the preferences of political actors in order to predict the outcome of majority rule choices in spatial models.

Suppose that the preferences of three political actors $N = \{A, B, C\}$ are arrayed as shown in Figure 1. The policy preferences of the three players are mapped at $\alpha = 1, .75,$ and $.50$ in two-dimensional policy space. The inner-most ellipse is the $\alpha = 1$ level for each player. The complete set of fuzzy alternatives are numbered and their corresponding three-tuple alpha-levels $(\tilde{A}(x), \tilde{B}(X), \tilde{C}(x))$, the intersection of the alpha-levels for the three players, noted in parentheses.

The set of options that are majority preferred (the winset) to each numbered alternative are noted in brackets. The set of undominated alternatives, whose winsets are empty, are in the maximal set and highlighted in gray in the figure. There are four alternatives in the maximal set circumscribed by this area: one uniting players B and C with a three-tuple of $(0, 1, .50)$ and three alternatives uniting one all three parties with three-tuples of $(.75, .75, .50), (.50, 1, .50),$ and $(.50, .75, .50)$.

3. Conditions under which Cycling Occurs in Fuzzy Spatial Models

While the foregoing example results in the prediction of a maximal set, such is not always the case. Figure 2 maps the fuzzy points at $\alpha = 1, .75, .50,$ and $.25$ for three players $A, B,$ and C . As before, we have labeled the three-tuple for each fuzzy point in the set of alternatives and identified the respective winsets in brackets. In this case, there is no maximal set. All points are majority preferred by at least one other point. However, alternatives 4, 16, and 27 comprise an

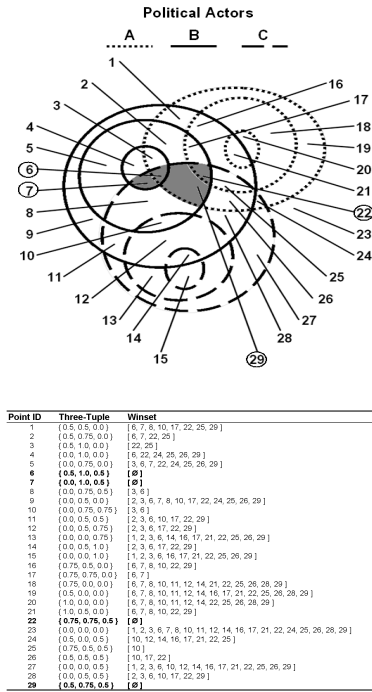


Figure 1: A Maximal Set in a Fuzzy Spatial Model

externally stable cycle. Alternative 4 is majority preferred to alternative 16, which is majority preferred to alternative 27, which is majority preferred to alternative 4. No alternative outside of this set is majority preferred to any alternative within it, and all alternatives outside of it are majority preferred by at least one point within it.

Hence, while the cycling problem appears to be substantially reduced in scope, the question of the extent to which the problem persists in fuzzy spatial models remains open. In what follows, we present a mathematical theorem that defines the conditions under which a fuzzy two-dimensional spatial model will produce a maximal set. The theorem makes it clear that it is not the shape of players' preferences that matters. For instance, the preferences in Figure 1 are convex while those in Figure 2 are non-convex. In fact, convex ellipses might result in an empty maximal set; and non-convex preferences can result in a maximal set. What is important are the intersections of the players' preferences. If the preferences for all three players intersect at relatively high α -levels or at α -level = 1 (the ideal region of alternatives) for any two of them, a maximal set always exists. Furthermore, our theorem demonstrates that fuzzy spatial models can map any geometric shape representing the preferences of players to a region with suitable and natural partial ordering. The results can be transferred faithfully back to the original spatial model. The formal proof of this transfer appears in the appendix. We summarize the conclusions in the final section of this paper.

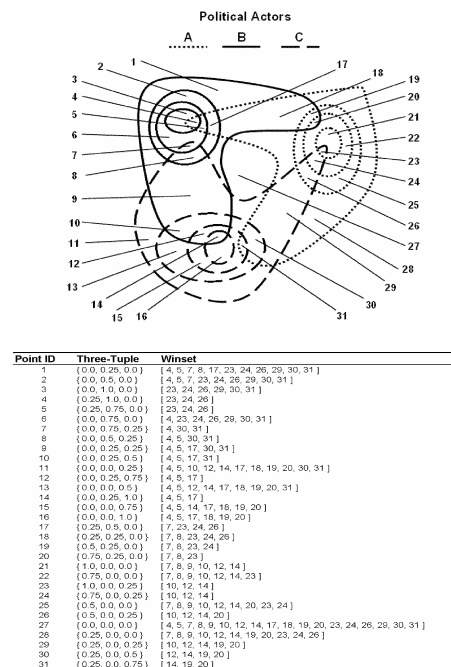


Figure 2: An Empty Maximal Set in a Fuzzy Spatial Model

3.1. Some Fundamentals

Let N denote the set of players and X denote a nonempty set of alternatives. We assume that X is a subset of a universe U of interest. In order to represent discrete fuzzy numbers of the type that we geometrically depicted in the previous discussion, we will let $T = \{0, .25, .5, .75, 1\}$ in our application. The results derived here are for arbitrary finite $T \subseteq [0, 1]$ with $0, 1 \in T$ and with U arbitrary. Later in this paper we demonstrate our results by applying them to the case $U = T^3$. We will be interested in fuzzy subsets of X with images from T . Let \mathbb{R}^+ denote the nonnegative real numbers and let \mathbb{R}^{+2} denote the set of ordered pairs $\mathbb{R}^+ \times \mathbb{R}^+$. Of primary interest to us in our application and subsequent demonstration are fuzzy subsets in $\mathcal{C} = \{ \tilde{X} : \mathbb{R}^{+2} \rightarrow T \mid 0, 1 \in \text{Im}(\tilde{X}), \tilde{X}^t \text{ is the interior and boundary of a simple closed curve } \forall t \in T \setminus \{0\} \}$, where $\text{Im}(\tilde{X})$ denotes the image of \tilde{X} . A simple closed curve is a curve for which there is a one-to-one open continuous function of the unit circle onto it. A simple closed curve has an interior that is bounded and an exterior. For notation, we use \vee to denote maximum, \setminus to denote set difference, and $||$ to denote the number of elements in a set.

We also have a particular interest in those fuzzy subsets \tilde{X} in \mathcal{C} for which \tilde{X}^t is a convex and compact set $\forall t \in T \setminus \{0\}$. Intuitively, a convex set is one for which the line segment joining any two points of the set lies in the set. A compact set is one that is closed and bounded.

Let \mathcal{R} denote the set of all binary relations on U that are reflexive, complete,

and transitive. Let $R_i \in \mathcal{R}$, $i \in N$. Then $xP_iy \Leftrightarrow xR_iy$ and not yR_ix . In such case, we say that x is **strictly preferred** to y by player i .

Let $\mathcal{R}^n = \{\boldsymbol{\rho} | \boldsymbol{\rho} = (R_1, \dots, R_n), R_i \in \mathcal{R}, i = 1, \dots, n\}$, where $|N| = n$. Let \leq be a partial order on U . The \leq relation is a relative unanimity partial ordering, where $xUy \Leftrightarrow xR_iy \forall i \in N$ and xP_iy for some $i \in N$. We assume in subsequent sections of this paper that \leq satisfies the following properties:

- (1) $\forall x, y \in U, x \leq y$ implies $\forall i \in N, yR_ix$;
- (2) $\forall x, y, z \in U, \forall i \in N, x \leq y$ and xR_iz implies yR_iz ;
- (3) $\forall x, y, z \in U, \forall i \in N, x \leq y$ and xP_iz implies yP_iz ;
- (4) $\forall x, y \in U, x < y$ implies $\exists i \in N$ such that yP_ix ;
- (5) $\forall x, y, z \in U, \forall i \in N, x \leq y$ and zR_iz implies zR_ix ;
- (6) $\forall x, y \in U, x$ and y incomparable under \leq and $\exists i \in N$ such that xP_iy implies $\exists j \in N$ such that yP_jx .

Let $\boldsymbol{\rho} \in \mathcal{R}^n$. Let f be an aggregation rule.

Definition 1.1. Define simple majority rule as follows. $\forall (x, y) \in X, (x, y) \in f(\boldsymbol{\rho})$ if and only if $|\{i \in N | xR_iy\}| > \frac{n}{2}$.

Then $(x, y) \in f(\boldsymbol{\rho})$ and not $(y, x) \in f(\boldsymbol{\rho}) \Leftrightarrow |\{i \in N | xR_iy\}| > \frac{n}{2}$ and $|\{i \in N | yR_ix\}| \leq \frac{n}{2} \Leftrightarrow |\{i \in N | xP_iy\}| > \frac{n}{2}$.

Definition 1.2. Define the binary relation R on X by $\forall x, y \in X, (x, y) \in R$ if and only if $|\{i \in N | xR_iy\}| \geq n/2$. Define $P \subseteq X \times X$ by $\forall x, y \in X, (x, y) \in P$

if and only if $(x, y) \in R$ and $(y, x) \notin R$. Let $R(x, y; \rho) = \{i \in N | xR_i y\}$ and $P(x, y; \rho) = \{i \in N | xP_i y\}$.

Proposition 1.1. Let $x, y \in X$. Then $(x, y) \in P$ if and only if $|P(x, y; \rho)| > n/2$.

Proof. $xPy \Leftrightarrow xRy$ and not $yRx \Leftrightarrow |\{i \in N | xR_i y\}| \geq n/2$ and $|\{j \in N | yR_j x\}| < n/2$. Since each R_i is complete, R is complete. Hence $xPy \Leftrightarrow |\{i \in N | xP_i y\}| > n/2$ by a simple counting procedure. Thus $xPy \Leftrightarrow |P(x, y; \rho)| > n/2$. ■

Definition 1.1. (Austen-Smith and Banks, 2005, p. 7) Define the **Pareto set** at $\tilde{R}, PS_N(\tilde{R})$, to be the set

$$PS_N(R) = \{x \in X | \forall y \in X (\exists i \in N, yP_i x \Rightarrow \exists j \in N, xP_j y)\}.$$

An alternative x is in the Pareto set if whenever a player strictly prefers an alternative w to x , then there is a player who strictly prefers x to w . Any effort by the group to choose other than an alternative in the Pareto set will leave at least one player worse off. Note that the Pareto set is not determined by majority rule but rather by unanimity.

3.2. The Existence of a Maximal Set

If we are to avoid cycling, then under majority rules, our models must predict a maximal set. Let N denote a set of players and \mathcal{A} denote an arbitrary set of

interest. We consider \mathcal{A} a set of alternatives. Our goal is to characterize the maximal set in \mathcal{A} with respect to a binary relation \tilde{R} in \mathcal{A} . We first characterize the maximal set for a set of alternatives X in a universe U for which there is a special function f^* of \mathcal{A} onto X such that the results determined in X can be carried back to corresponding results in \mathcal{A} . We first characterize the maximal set in X with respect to a relation R such that $f^*(\tilde{R}) = R$. In the appendix we prove that the characterized maximal set in X characterizes the maximal set in \mathcal{A} .

Definition 2.1. (Austen-Smith and Banks, 1999, p. 3) Let $M(R, X) = \{x \in X \mid xRy \forall y \in X\}$. Then $M(R, X)$ is called the **maximal set** of R .

We demonstrated earlier in this paper that a maximal set does not always exist in fuzzy spatial models. In what follows, we characterize the conditions under which that is the case. Our analysis begins with the Pareto set. As we will demonstrate, the characteristics of the elements in the Pareto set determine whether a maximal exists in fuzzy spatial models.

Definition 2.2. $L_R = \{x \in X \mid \nexists y \in X, x < y\}$. An element of L_R is called a **largest element** of X with respect to \leq . (Mathematics refers to an element in L_R , which is defined in terms of \leq , as a maximal element. We use “largest element” to avoid confusion with the elements comprising the maximal set given in Definition 2.1.) The following result shows the relationship of the set of maximal elements to the Pareto set.

Proposition 2.1. $L_R = PS_N(R)$.

Proof. Suppose $x \in L_R$. Let $y \in X$. Suppose $\exists i \in N$ such that $yP_i x$. Now there does not exist $y \in X$ such that $x < y$. Thus $\forall y \in X$, either $y \leq x$ or x and y are not comparable. Since $yP_i x, y \leq x$ is impossible else $xR_i y \forall i \in N$ by (1). Hence x and y are incomparable under \leq . Thus $\exists j \in N$ such that $xP_j y$ by property (6). Hence $x \in PS_N(R)$. Thus $L_R \subseteq PS_N(R)$.

Suppose $x \in PS_N(R)$. Suppose there exists $y \in X$ such that $x < y$. Then $\exists i \in N$ such that $yP_i x$. Since $x \in PS_N(R)$, there exists $j \in N$ such that $xP_j y$. Thus $x < y$ is impossible. Hence $x \in L_R$. Therefore $PS_N(R) \subseteq L_R$. ■

Corollary 2.1. Let $x \in X$.

(1) Suppose $\forall y \in X, x \leq y$ implies $x = y$. Then $x \in PS_N(R)$.

(2) If $x \notin PS_N(R)$, then there exists $y \in PS_N(R)$ such that $x < y$.

Proof. (1) Clearly $x \in L_R$, but $L_R = PS_N(R)$.

(2) Since $x \notin PS_N(R), x \notin L_R$. Thus there exists $y \in X$ such that $x < y$. Let y be the largest such element. Then $y \in L_R = PS_N(R)$. ■

Definition 2.3. Define $\langle \rangle : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ by $\forall S \in \mathcal{P}(U), \langle S \rangle = \{x \in U \mid \exists s \in S, x \leq s\}$.

Proposition 2.2. Let $\langle \rangle : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ be defined as above. Then the following conditions hold.

(1) $\forall S \in \mathcal{P}(U), S \subseteq \langle S \rangle$;

(2) $\forall S_1, S_2 \in \mathcal{P}(U), S_1 \subseteq S_2$ implies $\langle S_1 \rangle \subseteq \langle S_2 \rangle$;

$$(3) \forall S \in \mathcal{P}(U), \langle S \rangle = \langle \langle S \rangle \rangle;$$

$$(4) \forall S \in \mathcal{P}(U), \langle S \rangle = \cup_{s \in S} \langle \{s\} \rangle;$$

$$(5) \forall S \in \mathcal{P}(U), \forall x, y \in X, x \in \langle S \cup \{y\} \rangle \text{ and } x \notin \langle S \rangle \text{ implies } x \in \langle \{y\} \rangle.$$

Proof. (1) Let $s \in S$. Then $s \leq s$ and so $s \in \langle S \rangle$. Thus $S \subseteq \langle S \rangle$.

(2) Let $x \in \langle S_1 \rangle$. Then there exists $s \in S_1$ such that $x \leq s$. Since $s \in S_2, x \in \langle S_2 \rangle$.

(3) By (1), $\langle S \rangle \subseteq \langle \langle S \rangle \rangle$. Let $x \in \langle \langle S \rangle \rangle$. Then there exists $y \in \langle S \rangle$ such that $x \leq y$. There exists $s \in S$ such that $y \leq s$. Since \leq is transitive, $x \leq s$. Thus $x \in \langle S \rangle$. Hence $\langle \langle S \rangle \rangle \subseteq \langle S \rangle$.

(4) For all $s \in S, \langle \{s\} \rangle \subseteq \langle S \rangle$ by (2). Thus $\cup_{s \in S} \langle \{s\} \rangle \subseteq \langle S \rangle$. Let $x \in \langle S \rangle$. Then there exists $s \in S$ such that $x \leq s$. Thus $x \in \langle \{s\} \rangle$ and so $x \in \cup_{s \in S} \langle \{s\} \rangle$. Hence $\langle S \rangle \subseteq \cup_{s \in S} \langle \{s\} \rangle$.

(5) Suppose $x \in \langle S \cup \{y\} \rangle$ and $x \notin \langle S \rangle$. Then there does not exist $s \in S$ such that $x \leq s$. Hence $x \leq y$. Thus $x \in \langle \{y\} \rangle$. ■

The function $\langle \rangle$ is similar to that used to obtain structure results for (fuzzy) directed graphs, (fuzzy) finite state machines, and approximation spaces (Mordeson and Nair, 1996, Kuroki and Mordeson, 1997a, 1997b, Mordeson, 1999, Malik and Mordeson 2002). It may be possible to apply these structure results to $PS_N(R)$.

The following result is the *gateway* to our main conclusion. The result is critical in a series of lemmas that lead to our main conclusion (Theorem 2.2).

Theorem 2.1. $\langle X \rangle = \langle PS_N(R) \rangle$.

Proof. Clearly, $PS_N(R) \subseteq X$. Thus $\langle PS_N(R) \rangle \subseteq \langle X \rangle$. Let $x \in X$. If $x \notin \langle PS_N(R) \rangle$, then $x \notin PS_N(R)$ and so by (2) of Corollary 1.7, there exists $y \in PS_N(R)$ such that $x < y$. Thus $x \in \langle \{y\} \rangle \subseteq \langle PS_N(R) \rangle$. Hence $X \subseteq \langle PS_N(R) \rangle$ and so $\langle X \rangle \subseteq \langle PS_N(R) \rangle$. ■

The following lemma shows that if a maximal set exists, then at least one element of the maximal set is in the Pareto set.

Lemma 2.1. $M(R, X) \cap PS_N(R) = \emptyset$ if and only if $M(R, X) = \emptyset$.

Proof. Suppose $M(R, X) \neq \emptyset$. Let $x \in M(R, X)$. By Theorem 1.10, there exists $y \in PS_N(R)$ such that $y \geq x$. Since $x \in M(R, X)$, xRz for all $z \in X$ by property (3). Since $y \geq x$, yRz for all $z \in X$. Thus $y \in M(R, X)$. Hence $M(R, X) \cap PS_N(R) \neq \emptyset$. ■

The next lemma shows that an element in the Pareto set is in the maximal set of R if and only if it is not strictly preferred to (by majority rule) by another element in the Pareto set. Hence, the search for an element strictly preferred to another can be confined to the Pareto set.

Lemma 2.2. Let $s \in PS_N(R)$. Then there does not exist $c \in PS_N(R)$ such that cPs if and only if $s \in M(R, X)$.

Proof. Since R is complete and not cPs for all $c \in PS_N(R)$, it follows that sRc for all $c \in PS_N(R)$. Let $x \in X$. By Theorem 2.1, there exists $c \in PS_N(R)$ such that $c \geq x$. Thus sRx by property (5). Hence $s \in M(R, X)$. The converse is

immediate. ■

The next lemma establishes that if an element in the Pareto set can be majority defeated by any other element in X , it can be defeated by at least one element in the Pareto set.

Lemma 2.3. (1) Let $s \in PS_N(R)$. If there exists $x \in X$ such that xPs , then there exists $c \in PS_N(R)$ such that cPs .

(2) $M(R, X) = \emptyset$ if and only if $\forall s \in PS_N(R)$, there exists $c \in PS_N(R)$ such that cPs .

Proof. (1) By Theorem 2.1, there exists $c \in PS_N(R)$ such that $c \geq x$. Hence cPs by property (3).

(2) Suppose $M(R, X) = \emptyset$. Then the result holds by Lemma 1.12. Conversely, suppose $M(R, X) \neq \emptyset$. By Lemma 1.11, $M(R, X) \cap PS_N(R) \neq \emptyset$ and so there exists $s \in M(R, X) \cap PS_N(R)$. Hence there does not exist $c \in PS_N(R)$ such that cPs . ■

We can now state our main conclusion, which lays out the conditions under which a maximal set is empty in fuzzy spatial models.

Let $V = \{v \in U | v \text{ is not in a cycle}\}$. Let $N_1 = V \setminus N_2$, where $N_2 = \{w \in V | \forall R \in \mathcal{R}, w \in PS_N(R) \Rightarrow M(R, X) \neq \emptyset\}$. Let $M_1 = \{w \in V | \forall R \in \mathcal{R}^n, w \notin PS_N(R)\}$. Assume $M_1 \subseteq N_1$. Let $N'_1 = N_1 \setminus M_1$. Suppose N_1 is such that none of its elements are strictly preferred to one of $U \setminus V$.

Theorem 2.2. $M(R, X) = \emptyset$ if and only if $PS_N(R) = (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j) \cup N_1''$, where $N_1'' \subseteq N_1'$, C_k are cycles, $k = 1, \dots, n$, C'_j subsets of cycles which are not themselves cycles, $j = 1, \dots, m$, and

(1) $\forall s \in \cup_{j=1}^m C'_j$, there exists $c \in (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j)$ such that cPs ,

(2) $\forall s \in N_1''$ there exists $c \in (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j)$ such that cPs .

Proof. It follows that $PS_N(R) \subseteq (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j) \cup V$. Since no element of M_1 can be in $PS_N(R)$, $PS_N(R) \subseteq (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j) \cup (N_1 \setminus M_1) \cup N_2$. Hence it follows that $PS_N(R) = (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j) \cup N_1'' \cup N_2'$ for certain cycles $C_k, k = 1, \dots, n$, C'_j subsets of cycles which are not themselves cycles, $j = 1, \dots, m$, and for some $N_1'' \subseteq N_1'$, and $N_2' \subseteq N_2$.

Suppose $M(R, X) = \emptyset$. Since $N_2 \cap PS_N(R) \neq \emptyset$ implies $M(R, X) \neq \emptyset$, $PS_N(R) = (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j) \cup N_1''$, i. e., $N_2' = \emptyset$. Since no element of $N_1 \setminus M_1$ is preferred to one of $U \setminus V$, no element of $N_1 \setminus M_1$ is preferred to one of $PS_N(R)$. Hence $\forall s \in \cup_{j=1}^m C'_j, \exists c \in (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j)$ such that cPs by Lemma 2.2, else $M(R, X) \neq \emptyset$. By Lemma 2.2, $\forall s \in N_1''$, there exists $c \in (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j)$ such that cPs .

For the converse, the conditions imply $\forall s \in PS_N(R), \exists c \in PS_N(R)$ such that cPs . Hence no element of $PS_N(R)$ is in $M(R, X)$. Thus by Lemma 2.1, $M(R, X) = \emptyset$. ■

Theorem 2.3. If $PS_N(R)$ has no cycles, then $M(X, R) \neq \emptyset$.

Proof. Suppose that there are no strict preferences among the elements of

$PS_N(R)$. Then by Lemma 2.2, it follows that $PS_N(R) \subseteq M(R, X)$. Hence $M(R, X) \neq \emptyset$ in this case. Let $\bar{a}_1, \dots, \bar{a}_n$ be distinct elements of $PS_N(R)$ such that $\bar{a}_1 P \bar{a}_2, \bar{a}_2 P \bar{a}_3, \dots, \bar{a}_{n-1} P \bar{a}_n$ is of maximal length. If $n = 2$, then clearly it's not the case that $\bar{a}_1 P \bar{a}_1$ or $\bar{a}_2 P \bar{a}_1$. If $n \geq 3$, $\nexists \bar{a}_i$ such that $\bar{a}_i P \bar{a}_1$ else $\bar{a}_1 P \bar{a}_2, \dots, \bar{a}_{i-1} P \bar{a}_i$ is a cycle but $PS_N(R)$ has no cycles. By the maximality of n , $\nexists \bar{b} \in PS_N(R)$ distinct from $\bar{a}_i (i = 1, \dots, n)$ such that $\bar{b} P \bar{a}_1$. Hence no element of $PS_N(R)$ is strictly preferred to \bar{a}_1 . Thus \bar{a}_1 is a maximal element by Lemma 2.2. Hence $M(R, X) \neq \emptyset$. ■

3.3. A Three-Player Example

Our main conclusion is presented in Theorem 2.2. In all but a limited number of cases the maximal set is empty *if and only if* the Pareto set is a union of cycles or a subset of a union of cycles. We provide a formal proof in the appendix that this theorem and its attendant lemmas can be extended to the spatial models we presented in the earlier sections of this paper. Those models focused on a three-player game. In this section, we completely characterize the set of elements that constitute the exceptions to Theorem 2.2 in a three-player game. The characterized set is presented in Theorem 3.1.

Let the set of political players be $N = \{1, 2, 3\}$ and $i \in N$. Let $\tilde{A}_i : \mathcal{A} \rightarrow T$. Define the binary relation R_i on \mathcal{A} by for all $x, y \in \mathcal{A}, x R_i y$ if and only if $\tilde{A}_i(x) \geq \tilde{A}_i(y)$. In such a case, we say that x is at least as good as y for player i . Clearly, R_i is reflexive, complete, and transitive. Now $x P_i y \Leftrightarrow x R_i y$ and not

$yR_ix \Leftrightarrow \tilde{A}_i(x) > \tilde{A}_i(y)$. In such case, we say that x is **strictly preferred** to y by player i .

We now consider our application area. Let the set of players $N = \{1, 2, 3\}$ and associate the fuzzy subsets \tilde{A}_i of \mathcal{A} with i for $i = 1, 2, 3$. Let R_i be defined in terms of \tilde{A}_i as above, $i = 1, 2, 3$. Let \mathcal{R} denote the set of all reflexive, complete, and transitive binary relations on \mathcal{A} ; \mathcal{R}^3 the set of all ordered triples of elements of \mathcal{R} ; and \mathcal{B} the set of all reflexive and complete binary relations on \mathcal{R} . Define simple majority rule $f : \mathcal{R}^3 \rightarrow \mathcal{B}$ as follows: $\forall \rho = (R_1, R_2, R_3), \forall (x, y) \in \mathcal{A} \times \mathcal{A}, (x, y) \in f(\rho)$ if and only if $|\{i \in N | xR_i y\}| \geq 2$. Then $(x, y) \in f(\rho)$ and not $(y, x) \in f(\rho) \Leftrightarrow |\{i \in N | xR_i y\}| \geq 2$ and $|\{i \in N | yR_i x\}| \leq 1 \Leftrightarrow |\{i \in N | xP_i y\}| \geq 2$. Let $\tilde{R} = f(\rho)$.

Definition 3.1. (Austen-Smith and Banks, 2005, p. 7) Define the **Pareto set** at $\tilde{R}, PS_N(\tilde{R})$, to be the set

$$PS_N(\tilde{R}) = \{x \in \mathcal{A} | \forall w \in \mathcal{A}, w \neq x, \exists i \in N, wP_i x \Rightarrow \exists j \in N, xP_j w\}.$$

An alternative x is in the Pareto set if whenever a player strictly prefers an alternative w to x , then there is a player who strictly prefers x to w . Any effort by the group to choose other than an alternative in the Pareto set will leave at least one player worse off. Note that the Pareto set is not determined by majority rule but rather by unanimity. Clearly, $PS_N(\tilde{R}) = \{x \in \mathcal{A} | \forall w \in \mathcal{A}, w \neq x, \exists i \in N, \tilde{A}_i(w) > \tilde{A}_i(x) \Rightarrow \exists j \in N, \tilde{A}_j(x) > \tilde{A}_j(w)\}$.

Let T^3 denote the set of all ordered triples of elements from T .

Definition 3.2. Define $f^* : \mathcal{A} \rightarrow T^3$ by $\forall x \in \mathcal{A}, f^*(x) = (r, s, t)$, where

$$r = \vee\{a \in T \mid x \in \tilde{A}_1^a\}, s = \vee\{b \in T \mid x \in \tilde{A}_1^b\}, t = \vee\{c \in T \mid x \in \tilde{A}_3^c\}.$$

Then (r, s, t) is called the **assignment** of x with respect to $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3$. An element (a, b, c) of T^3 is called **allowable** with respect to $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3$ if there exists $x \in \mathcal{A}$ such that $f^*(x) = (a, b, c)$. Let X denote the set of all allowable elements of T^3 with respect to $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3$.

In Definition 3.2, it is clear that $x \in \tilde{A}_1^r \cap \tilde{A}_2^s \cap \tilde{A}_3^t$. It is also clear that for $r', s', t' \in T$ with $r \leq r', s \leq s', t \leq t'$, it is not the case that $x \in \tilde{A}_1^{r'} \cap \tilde{A}_2^{s'} \cap \tilde{A}_3^{t'}$ if any of the inequalities are strict.

The following results show that the main results, which are determined in T^3 , can be transferred faithfully to \mathcal{A} via the function f^* of \mathcal{A} into T^3 .

Definition 3.3. Define the binary relation R on T^3 as follows: $\forall (a, b, c), (d, e, f) \in T^3, (a, b, c)R(d, e, f)$ if and only if either $a \geq d, b \geq e$ or $a \geq d, c \geq f$ or $b \geq e, c \geq f$.

Define the binary relation P on T^3 by $\forall (a, b, c), (d, e, f) \in T^3,$

$$(a, b, c)P(d, e, f) \text{ if and only if } (a, b, c)R(d, e, f) \text{ and not } (d, e, f)R(a, b, c).$$

Proposition 3.1. Let R and P be defined as in Definition 3.3. Then $\forall (a, b, c), (d, e, f) \in$

T^3 ,

$(a, b, c)P(d, e, f)$ if and only if $a > d, b > e$ or $a > d, c > f$ or $b > e, c > f$.

Proof. It follows easily that R is complete. Thus $\forall (a, b, c), (d, e, f) \in T^3, (a, b, c)P(d, e, f)$ if and only if not $(d, e, f)R(a, b, c)$. Now not $(d, e, f)R(a, b, c) \Leftrightarrow$ not $(d \geq a, e \geq b$ or $d \geq a, f \geq c$ or $e \geq b, f \geq c) \Leftrightarrow$ (not $(d \geq a, e \geq b)$ and not $(d \geq a, f \geq c)$ and not $(e \geq b, f \geq c)) \Leftrightarrow$ (not $d \geq a$ or not $e \geq b$) and (not $d \geq a$ or not $f \geq c$) and (not $e \geq b$ or not $f \geq c) \Leftrightarrow ((a > d$ or $b > e)$ and $(a > d$ or $c > f)$ and $(b > e$ or $c > f)) \Leftrightarrow a > d, b > e$ or $a > d, c > f$ or $b > e, c > f$. Hence the desired result holds. ■

Definition 3.4. Define the binary relation \widehat{R} on \mathcal{A} as follows: $\forall x, y \in \mathcal{A}, x\widehat{R}y$ if and only if $(a, b, c)R(d, e, f)$, where (a, b, c) and (d, e, f) are the assignments of x and y , respectively, with respect to $\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3$. Define the binary relation \widehat{P} on \mathcal{A} by $\forall x, y \in \mathcal{A}, x\widehat{P}y$ if and only if $x\widehat{R}y$ and not $y\widehat{R}x$.

The following result is an easy consequence of the definitions. It makes the connection between strict preference in \mathcal{A} and strict preference in T^3 .

Proposition 3.2. For all $x, y \in \mathcal{A}, x\widehat{P}y$ if and only if $(a, b, c)P(d, e, f)$, where (a, b, c) and (d, e, f) are the assignments of x and y , respectively, with respect to $\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3$.

Let $x, y \in \mathcal{A}$. Let $(a, b, c) = f^*(x)$ and $(d, e, f) = f^*(y)$. Then $x\widehat{R}y \Leftrightarrow$

$f^*(x)Rf^*(y) \Leftrightarrow (a, b, c)R(d, e, f) \Leftrightarrow \tilde{A}_1(x) \geq \tilde{A}_1(y), \tilde{A}_2(x) \geq \tilde{A}_2(y)$ or $\tilde{A}_1(x) \geq \tilde{A}_1(y), \tilde{A}_3(x) \geq \tilde{A}_3(y)$ or $\tilde{A}_2(x) \geq \tilde{A}_2(y), \tilde{A}_3(x) \geq \tilde{A}_3(y) \Leftrightarrow xR_1y, xR_2y$ or xR_1y, xR_3y or $xR_2y, xR_3y \Leftrightarrow x\tilde{R}y$. Thus \hat{R} and \tilde{R} are the equivalent. (This argument shows that f^* is a homomorphism as defined in the appendix.) It also follows that

$$PS_N(R) = \{(a, b, c) \in f^*(\mathcal{A}) | \forall (d, e, f) \in f^*(\mathcal{A}), a < d \Rightarrow b > e \text{ or } c > f$$

$$\text{and } b < e \Rightarrow a > d \text{ or } c > f \text{ and } c < f \Rightarrow a > d \text{ or } b > e\}.$$

Thus we see that the preimage of $PS_N(R)$ under f^* equals $PS_N(\tilde{R})$.

Let $U = T^3$, where $T = \{0, .25, .5, .75, 1\}$. The partial order \leq on T^3 is defined by $\forall (a_1, a_2, a_3), (b_1, b_2, b_3) \in T^3, (a_1, a_2, a_3) \leq (b_1, b_2, b_3)$ if and only if $a_i \leq b_i$ for $i = 1, 2, 3$. Then $(a_1, a_2, a_3) < (b_1, b_2, b_3)$ if and only if $(a_1, a_2, a_3) \leq (b_1, b_2, b_3)$ and $a_i < b_i$ for some $i = 1, 2, 3$. For conditions (1) – (6) in Section 3.1, the relations R_i are defined as follows: $\forall (a_1, a_2, a_3), (b_1, b_2, b_3) \in T^3, (a_1, a_2, a_3)R_i(b_1, b_2, b_3)$ if and only if $a_i \geq b_i, i = 1, 2, 3$. It follows easily that conditions (1) – (6) hold. (Suppose $f^* : \mathcal{A} \rightarrow T^3$ is such that for $x, y \in \mathcal{A}, f_i^*(x) = (a_1, a_2, a_3)$ and $f_i^*(y) = (b_1, b_2, b_3)$. Then for $i = 1, 2, 3, f_i^*(x)R_i f_i^*(y) \Leftrightarrow (a_1, a_2, a_3)R_i(b_1, b_2, b_3) \Leftrightarrow a_i \geq b_i \Leftrightarrow \tilde{A}_i(x) \geq \tilde{A}_i(y) \Leftrightarrow x\tilde{R}_i y$. Thus f_i^* satisfies the properties in Theorem A.3 of the appendix.) Hence the results of Sections 1 and 2 hold here.

Lemma 3.1. Let

$$N_1 = \{(a, b, c) | a, b, c \in \{0, .25\}\} \cup \{(0, 0, a) | a \in T\} \cup \{(0, a, 0) | a \in T\} \cup \{(a, 0, 0) | a \in T\}$$

and

$$N_2 = \{(d, e, f) | d, e, f \in \{.75, 1\}\} \cup \{(1, 1, a) | a \in T\} \cup \{(1, a, 1) | a \in T\} \cup \{(a, 1, 1) | a \in T\}.$$

Let $(x, y, z) \in T^3$. Then there does not exist a cycle in T^3 containing (x, y, z) if and only if $(x, y, z) \in N_1 \cup N_2$.

Proof. Let $\bar{x} \in N_2$. If $\bar{y} \in T^3$ such that $\bar{y}P\bar{x}$, then \bar{x} is not in a cycle. If there exists $\bar{y} \in T^3$ such that $\bar{y}P\bar{x}$, then two of the components of \bar{y} equal 1. Thus there does not exist $\bar{z} \in T^3$ such that $\bar{z}P\bar{y}$. Hence \bar{x} is not in a cycle. Let $\bar{x} \in N_1$. If $\bar{y} \in T^3$ such that $\bar{x}P\bar{y}$, then \bar{x} is not in a cycle. If there exists $\bar{y} \in T^3$ such that $\bar{x}P\bar{y}$, then two components of \bar{y} equal 0. Hence there does not exist $\bar{z} \in T^3$ such that $\bar{y}P\bar{z}$. Thus \bar{x} is not in a cycle.

We complete the proof by showing any other element of T^3 is in a cycle. Consider (a, b, c) , where a, b, c are pairwise distinct, say $a < b < c$. Then $\{(c, a, b), (b, c, a), (a, b, c)\}$ is a cycle. Suppose $a = b = c$. Then the previous paragraph shows that $a = b = c = .5$. Clearly $\{(.75, 0, .75), (.5, .5, .5), (.25, .25, 1)\}$ is a cycle. Suppose exactly two of a, b, c are equal, say $a = b$. There are 10 remaining elements (a, a, c) and their

permutations $(a, c, a), (c, a, a)$. It suffices to consider the 10 elements,

$$\begin{aligned} & (.5, .5, 0), (.5, .5, .25), (.5, .5, .75), (.5, .5, 1), \\ & (.25, .25, .5), (.25, .25, .75), (.25, .25, 1), \\ & (.75, .75, 0), (.75, .75, .25), (.75, .75, .5). \end{aligned}$$

The following are cycles involving the 10 elements or permutations of the elements:

$$\begin{aligned} & \{(0, .75, .75), (.5, .5, 0), (.25, .25, 1)\}, \{(0, .75, .5), (.5, .5, .25), (.25, .25, .75)\}, \\ & \{(0, .75, 1), (.5, .5, .75), (.25, 1, .25)\}, \{(.75, .75, c), (.5, .5, 1), (.25, 1, .75)\}, \\ & \{(.5, .5, 0), (.25, .25, .5), (.75, 0, .25)\}, \end{aligned}$$

where $c = 0, .25$ or $.5$. ■

Let $N'_1 = \{(.25, .25, .25), (.25, .25, 0), (.25, 0, .25), (0, .25, .25)\}$ and $I_1 = \{(1, 0, 0), (0, 0, 1), (0, 1, 0)\}$.

We can now state our main conclusion, which lays out the conditions under which a maximal set is empty in fuzzy three-player spatial models.

Theorem 3.1. $M(R, X) = \emptyset$ if and only if $PS_N(R) = (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j) \cup N''_1 \cup I'_1$, where $N''_1 \subseteq N'_1, I'_1 \subseteq I_1, C_k$ are cycles, $k = 1, \dots, n, C'_j$ are subsets of cycles which are not themselves cycles, $j = 1, \dots, m$, and

- (1) $\forall \bar{s} \in \cup_{j=1}^m C'_j$, there exists $\bar{c} \in (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j)$ such that $\bar{c}P\bar{s}$,
- (2) $\forall \bar{s} \in N''_1$ there exists $\bar{c} \in (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j)$ such that $\bar{c}P\bar{s}$ and

(3) $\forall \bar{i} \in I'_1, \exists \bar{d} \in (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j) \cup N''_1$ such that $\bar{d}P\bar{i}$.

Proof. By Lemma 3.1, it follows that $PS_N(R) \subseteq (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j) \cup N_1 \cup N_2$. Since no element of $(\{(0, 0, a) | a \in T\} \cup \{(0, a, 0) | a \in T\} \cup \{(a, 0, 0) | a \in T\}) \setminus I_1$ can be in $PS_N(R)$ and since no element of $N_1 \setminus (N'_1 \cup I_1)$ can be in $PS_N(R)$, $PS_N(R) \subseteq (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j) \cup N'_1 \cup I_1 \cup N_2$. Hence it follows that $PS_N(R) = (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j) \cup N''_1 \cup I'_1 \cup N'_2$ for certain cycles $C_k, k = 1, \dots, n, C'_j$ subsets of cycles which are not themselves cycles, $j = 1, \dots, m$, and for some $N''_1 \subseteq N'_1, I'_1 \subseteq I_1$, and $N'_2 \subseteq N_2$.

Suppose $M(R, X) = \emptyset$. By Lemma 2.1, $PS_N(R) = (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j) \cup N''_1 \cup I'_1$, i. e., $N'_2 = \emptyset$. If $\bar{s} \in N_1$ is such that $\bar{s}P\bar{c}$ for some $\bar{c} \in PS_N(R)$, then two of the components of \bar{c} equal 0 and so $\bar{c} \in I'_1$. Thus no element of N_1 is strictly preferred to one of $(\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j)$. No element of I'_1 is strictly preferred to any element of $PS_N(R)$. Hence $\forall \bar{s} \in \cup_{j=1}^m C'_j, \exists \bar{c} \in (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j)$ such that $\bar{c}P\bar{s}$ by Lemma 2.2 else $M(R, X) \neq \emptyset$. By Lemma 2.2, $\forall \bar{s} \in N''_1$ there exists $\bar{c} \in (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j)$ such that $\bar{c}P\bar{s}$ and $\forall \bar{i} \in I'_1, \exists \bar{d} \in (\cup_{k=1}^n C_k) \cup (\cup_{j=1}^m C'_j) \cup N''_1$ such that $\bar{d}P\bar{i}$.

For the converse, the conditions imply $\forall \bar{s} \in PS_N(R), \exists \bar{c} \in PS_N(R)$ such that $\bar{c}P\bar{s}$. Hence no element of $PS_N(R)$ is in $M(X, R)$. Thus by Lemma 2.3, $M(R, X) = \emptyset$. ■

We can have $(\cup_{k=1}^n C_k) \cap (\cup_{j=1}^m C'_j) = \emptyset$ in Theorem 3.1 by deleting any element $\bar{s} \in (\cup_{k=1}^n C_k) \cap (\cup_{j=1}^m C'_j)$ from $\cup_{j=1}^m C'_j$ since \bar{s} is in some C_k . Also any elements

of $\cup_{j=1}^m C'_j$ that form a cycle can be removed from $\cup_{j=1}^m C'_j$ and moved to $\cup_{k=1}^n C_k$ since they form a cycle.

Example 3.1. Let $PS_N(R) = C \cup \{(.75, .75, .75)\}$, where $C = \{(1, .5, 0), (.5, 0, 1), (0, 1, .5)\}$. Then C is a cycle. Thus $M(R, X) \cap PS_N(R) = \{(.75, .75, .75)\}$. It is easily verified that $M(R, X) = \{(.75, .75, .75, (.75, .75, .5), (.75, .5, .75), (.5, .75, .75)\}$. Here $M(R, X) \not\subseteq PS_N(R)$.

Example 3.2. Let $PS_N(R) = C \cup \{(.75, .75, .75)\}$, where $C = \{(1, .75, 0), (.75, 0, 1), (0, 1, .75)\}$. Then C is a cycle. Thus $M(R, X) \cap PS_N(R) = \{(.75, .75, .75)\}$. It is easily verified that $M(R, X) = \{(.75, .75, .75)\}$. In this example, $M(R, X) \subseteq PS_N(R)$.

4. Implications

Under an assumption of sincere voting, crisp spatial models can only avoid cycling when Plott's (1967) radial symmetry obtains. If sophisticated voting is assumed, crisp spatial models must resort to either the uncovered set or imposition of a set of highly restrictive assumptions. Fuzzy spatial models can predict a maximal set under far less restrictive conditions. They can also deal with irregular shaped preferences, which standard mathematical approaches can only do with substantial difficulty. Theorem 2.2 establishes that in all but a limited number of cases the maximal set is empty *if and only if* the Pareto set is a union of cycles or a subset of a union of cycles. Hence, if the elements in the Pareto set cycle

under majority rule or if they constitute a subset of a cycle under majority rule, the maximal set is empty; and vice versa. Moreover, Theorem 2.3 establishes that for the maximal set to be empty, the Pareto set must contain at least one *complete* cycle.

Furthermore, we have demonstrated that if the maximal set exists, at least one of its elements must be contained in the Pareto set (Lemma 2.1). If an element in the Pareto set can not be majority defeated by any other element in the Pareto set, then it is an element in the maximal set. However, if every element in the Pareto set can be majority defeated by at least one other element in the Pareto set, then the maximal set is empty. Moreover, if an element in the Pareto set can be majority defeated by any element in the total set of available alternatives then at least one of the elements that defeats it must be in the Pareto set (Lemma 2.3). The implication is that we may confine our initial search for the existence of a maximal set to the Pareto set. Moreover, Theorem 2.1 makes it clear that the elements in the Pareto set that are also contained in the maximal set define any remaining elements in the maximal set.

Theorem 2.2 can be applied to any n -player game ($n > 2$). Theorem 3.1 completely characterizes the limited cases that constitute the exceptions for a three-player game. While none of the following seven distinct elements are ever part of any cycle, if they are part of the Pareto set together with a cycle or a subset of a cycle, and they are defeated by at least one other element in the Pareto set,

then the maximal set is empty.

$$\{(.25, .25, .25), (.25, .25, 0), (.25, 0, .25), (0, .25, .25), \\ \{(1, 0, 0), (0, 0, 1), (0, 1, 0)\}.$$

If they are not defeated by at least one other element in the Pareto set or if they uniquely comprise the Pareto set, then they are elements in the maximal set. The first four elements represent a situation in which the players' preferences intersect at the lowest α -level possible. In the most trivial case, represented by $\{(1, 0, 0), (0, 0, 1), (0, 1, 0)\}$, the preferences of three players do not intersect at any α -level. In this case, the Pareto set and maximal set comprise the same three alternatives (the "ideal" points of the players).

Furthermore, if any of the following elements are in the Pareto set, a maximal set always exists:

$$\{(1, 1, a), (1, a, 1), (a, 1, 1), (1, .75, .75), (.75, 1, .75), (.75, .75, 1), \\ (.75, .75, .75)\}, \text{ where } a \text{ is any element of } T.$$

In essence, if the preferences for all three players intersect at relatively high

α -levels or at α -level = 1 (the ideal region of alternatives) for any two of them, a maximal set exists.

The fuzzy approach commends itself for use in a great many spatial models in political science. In fact, the assumptions of fuzzy set theory are far closer to the human reality than those of the conventional crisp set approach. Fuzzy set theory permits models to deal with a substantial degree of indifference. This indifference over a large set of options is not related to the uncertainty that actors have over their own ideal position or those of others. A number of approaches have already been developed that permit researchers to deal with incomplete information. Rather, it concerns political actors' *ambiguity* about their ideal policy positions. Actors may be certain of the general area within which their policy positions lie without being able to identify an exact location. Such ambiguity or vagueness, as opposed to uncertainty, seems to us to be far more likely in politics than a strict assumption that reduces the set of ideal policy positions to a single alternative. When such is the case, scholars should consider using a fuzzy approach to spatial modeling.

5. Appendix

We demonstrate that our formal proofs are relevant to spatial models by carrying the results determined in n -player games back to \mathbb{R}^{+2} . We accomplish this by defining an appropriate function f^* from \mathcal{A} into X . (The results obtained can be applied to T^n where n denotes the number of players.) Let \mathcal{A} denote the region of interest and X the region in which \mathcal{A} is mapped by the homomorphism f^* . The following formal discussion shows that the results determined in X concerning the maximal set can be transferred faithfully back to \mathcal{A} .

Let \tilde{R} be a binary relation on a set \mathcal{A} . The pair (\mathcal{A}, \tilde{R}) is called a **relation space**.

Definition A.1. Let (\mathcal{A}, \tilde{R}) and (X, R) be relation spaces. Let f^* be a function of \mathcal{A} into X . Then f^* is called a **homomorphism** of (\mathcal{A}, \tilde{R}) into (X, R) if $\forall a, b \in \mathcal{A}, (a, b) \in \tilde{R}$ if and only if $(f^*(a), f^*(b)) \in R$. If f^* maps \mathcal{A} onto X , we say f^* maps (\mathcal{A}, \tilde{R}) onto (X, R) . For all $(a, b) \in \tilde{R}$, we write $f^*((a, b)) = (f^*(a), f^*(b))$ and $f^*(\tilde{R}) = \{(f^*((a, b)) | (a, b) \in \tilde{R}\}$.

Let f^* be a **homomorphism** of (\mathcal{A}, \tilde{R}) into (X, R) . Then $\forall a, b \in \mathcal{A}, (a, b) \in \tilde{R}$ if and only if $(f^*(a), f^*(b)) \in R$. Thus if $a, a', b, b' \in \mathcal{A}$ and $f^*(a) = f^*(a'), f^*(b) = f^*(b')$, it is not possible that $(a, b) \in \tilde{R}$ and $(a', b') \notin \tilde{R}$.

Proposition A.1. Let f^* be a homomorphism of (\mathcal{A}, \tilde{R}) onto (X, R) . Then $f^*(\tilde{R}) = R$.

Proof. Clearly, $f^*(\tilde{R}) \subseteq R$. Let $(x, y) \in R$. Since f^* maps \mathcal{A} onto X , there exists $a, b \in \mathcal{A}$ such that $f^*(a) = x$ and $f^*(b) = y$. Thus $(x, y) = (f^*(a), f^*(b)) = f^*((a, b)) \in f^*(\tilde{R})$. ■

Proposition A.2. Let f^* be a homomorphism of (\mathcal{A}, \tilde{R}) onto (X, R) . Then $\forall a, b \in \mathcal{A}, (a, b) \in \tilde{P}$ if and only if $(f^*(a), f^*(b)) \in P$.

Proof. Let $a, b \in \mathcal{A}$. Then $(a, b) \in \tilde{P} \Leftrightarrow (a, b) \in \tilde{R}, (b, a) \notin \tilde{R} \Leftrightarrow (f^*(a), f^*(b)) \in R, (f^*(a), f^*(b)) \notin R \Leftrightarrow (f^*(a), f^*(b)) \in P$. ■

Theorem A.1. Let f^* be a homomorphism of (\mathcal{A}, \tilde{R}) onto (X, R) . Then $f^*(M(\tilde{R}, \mathcal{A})) = M(R, X)$.

Proof. $a \in M(\tilde{R}, \mathcal{A}) \Leftrightarrow \forall b \in \mathcal{A}, a\tilde{R}b \Leftrightarrow \forall f^*(b) \in X, f^*(a)Rf^*(b) \Leftrightarrow f^*(a) \in M(R, X)$, where the latter equivalence holds since f^* maps \mathcal{A} onto X . Thus if $f^*(a) \in f^*(M(\tilde{R}, \mathcal{A}))$, then $a \in M(\tilde{R}, \mathcal{A})$. Hence $f^*(a) \in M(R, X)$. Thus $f^*(M(\tilde{R}, \mathcal{A})) \subseteq M(R, X)$. Let $x \in M(R, X)$. Then $\forall y \in X, xRy$. Let $a \in \mathcal{A}$ be such that $f^*(a) = x$. Let $b \in \mathcal{A}$. Then $f^*(a)Rf^*(b)$ since $x = f^*(a)$ and $x \in M(R, X)$. Hence $a\tilde{R}b$ by Definition A.1. Thus $a \in M(\tilde{R}, \mathcal{A})$ and so $x = f^*(a) \in f^*(M(\tilde{R}, \mathcal{A}))$. Thus $M(R, X) \subseteq f^*(M(\tilde{R}, \mathcal{A}))$.

Let $(\mathcal{A}, \tilde{R}_i)$ be a relation space, $i = 1, \dots, n$. Let f_i^* be a homomorphism of $(\mathcal{A}, \tilde{R}_i)$ onto $(X, R_i), i = 1, \dots, n$. Then $R_i = f_i^*(\tilde{R}_i), i = 1, \dots, n$ by Proposition A.1. ■

Definition A.2. Let \tilde{f} be an aggregation rule on $(\mathcal{A}, (\tilde{R}_1, \dots, \tilde{R}_n))$ and let f

be an aggregation rule on $(X, (R_1, \dots, R_n))$. Let f_i^* be a homomorphism of $(\mathcal{A}, \tilde{R}_i)$ onto (X, R_i) , $i = 1, \dots, n$. Let f^* be a homomorphism of $(\mathcal{A}, \tilde{f}((\tilde{R}_1, \dots, \tilde{R}_n)))$ onto $(X, f((R_1, \dots, R_n)))$. Then f^* is said to **preserve the pair (\tilde{f}, f) with respect to (f_1^*, \dots, f_n^*)** if $f^*(\tilde{f}(\tilde{R}_1, \dots, \tilde{R}_n)) = f((R_1, \dots, R_n))$.

Theorem A.2. Let \tilde{f} be an aggregation rule on $(\mathcal{A}, (\tilde{R}_1, \dots, \tilde{R}_n))$ and let f be an aggregation rule on $(X, (R_1, \dots, R_n))$. Let f_i^* be a homomorphism of $(\mathcal{A}, \tilde{R}_i)$ onto (X, R_i) , $i = 1, \dots, n$. Let f^* be a homomorphism of $(\mathcal{A}, \tilde{f}((\tilde{R}_1, \dots, \tilde{R}_n)))$ onto $(X, f((R_1, \dots, R_n)))$ such that f^* preserves (\tilde{f}, f) w.r.t. (f_1^*, \dots, f_n^*) . Then $f^*(PS_N(\tilde{R})) = PS_N(R)$, where $\tilde{R} = \tilde{f}((\tilde{R}_1, \dots, \tilde{R}_n))$ and $R = f((R_1, \dots, R_n))$.

Proof. $a \in PS_N(\tilde{R}) \Leftrightarrow \forall b \in \mathcal{A}, (\exists i \in N, b\tilde{P}_i a \Rightarrow \exists j \in N, a\tilde{P}_j b)$. Suppose $f^*(a) \in f^*(PS_N(\tilde{R}))$. Then $a \in PS_N(\tilde{R})$ by definition of a homomorphism. Hence $f^*(a) \in PS_N(R)$. Thus $f^*(PS_N(\tilde{R})) \subseteq PS_N(R)$. Let $x \in PS_N(R)$. Let $y \in X$. If $\exists i \in N$ such that $yP_i x$, then $\exists j \in N$ such that $xP_j y$. Let $a \in \mathcal{A}$ be such that $f^*(a) = x$. Let $b \in \mathcal{A}$. Then $f^*(b)P_i f^*(a) \Leftrightarrow b\tilde{P}_i a$ and $f^*(a)P_j f^*(b) \Leftrightarrow a\tilde{P}_j b$. Thus if $\exists i \in N$ such that $b\tilde{P}_i a$, then $\exists j \in N$ such that $a\tilde{P}_j b$. Thus $a \in PS_N(\tilde{R})$ and so $x = f^*(a) \in f^*(PS_N(\tilde{R}))$. Hence $PS_N(R) \subseteq f^*(PS_N(\tilde{R}))$. ■

Theorem A.3. Let \tilde{f} be an aggregation rule on $(\mathcal{A}, (\tilde{R}_1, \dots, \tilde{R}_n))$ and let f be an aggregation rule on $(X, (R_1, \dots, R_n))$. Let f_i^* be a homomorphism of $(\mathcal{A}, \tilde{R}_i)$ onto (X, R_i) , $i = 1, \dots, n$. Let f^* be a homomorphism of $(\mathcal{A}, \tilde{f}((\tilde{R}_1, \dots, \tilde{R}_n)))$ onto $(X, f((R_1, \dots, R_n)))$ such that f^* preserves (\tilde{f}, f) w.r.t. (f_1^*, \dots, f_n^*) . If \tilde{f} is a simple

majority rule, then f is a simple majority rule.

Proof. Since by Proposition A.2, $\forall a, b \in \mathcal{A}$, $(a, b) \in \tilde{P}_i$ if and only if $(f^*(a), f^*(b)) \in P_i$, $i = 1, \dots, n$, it follows that $|\tilde{P}(a, b; \tilde{f}((\tilde{R}_1, \dots, \tilde{R}_n)))| = |P(f^*(a), f^*(b)); f((R_1, \dots, R_n))|$.

The desired result now follows. ■

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