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[Reprinted from THE AMERICAN MATHEMATICAL MONTHLY, Vol. XXVI, October, 1919.]

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The rose, rosette, rosace, Rosenkurve or multifolium, or whatever other name it may have, is a periodic polar curve whose equal sectors may have any angular magnitude. Its general equation, as usually given, $\rho = a + b \sin n\theta$,[2] supposes the tracing point to move with a simple harmonic motion of n cycles along a radial line through the pole, at the same time that it makes one revolution about this pole with uniform angular speed. A simple instance of such a polar curve is the trifolium, Fig. 1 (p. 324), which is drawn by having a tracing pen move with simple harmonic motion of amplitude b in a radial line over a uniformly rotating disk in such a way that the pen just touches its centre without passing beyond, and the disk makes one revolution in three cycles of the pen. The equation is then $\rho = a(1 - \sin 3\theta)$, as is seen by inspection in Fig. 2.

If in the general equation a is greater than b, the pen does not reach the center as in Fig. 3, and if a is less than b, the pen passes beyond it as in Fig. 4 and draws

[1] See "Concerning a new Method of Tracing Cardioids" by William F. Rigge, in the January, 1919, MONTHLY and "On the Construction of Certain Curves Given in Polar Coördinates" by R. E. Moritz, in the May, 1917, MONTHLY.

[2] The two terms may have unlike signs and sin be replaced by cos.

smaller lobes or loops on the other side, which are within or between the larger ones according as n is odd or even, until when $a = 0$, the equation becomes $\rho = b \sin 3\theta$, the loops are equal and we have a trifolium in Fig. 5 somewhat like Fig. 1 but half its size and traced twice by the pen because n is odd, the number of the lobes being doubled when n is even. The rosette is *cuspidal* only when $a = b$ as in Figs. 1 and 2, because only then the pen, after tracing one branch of the curve and coming to a momentary standstill, retreats along another branch, so that both branches have a common rectilinear tangent at the point of rest. This tangent is always between the two branches in the curves treated in this discussion, so that the cusps are of the first species.

All this is well known and has been mentioned merely as an introduction to the subject in hand. As Moritz (l. c.) has very completely treated the case of the pen moving along a *radial* line through the center of the disk, the problem now is to investigate what happens when the pen moves along a *non-radial* line, and when it is set down at any initial phase. One who has tried the experiment will know that he obtained a rosette distorted and somewhat like one of the three mentioned before in Figs. 3, 2 and 4, which we may call rounded or curtate, cuspidal, and looped or prolate. Of these the cuspidal rosettes require certain conditions which it is the specific object of the present article to study.

The Cardioid.—As the cardioid may be defined as a cuspidal rosette with the ratio $n = 1$, the present investigations are based upon the previously cited article by the writer, to which the reader must have recourse for more detailed explanations. The following condensed resumé may however be sufficient.

In Fig. 6, A is the center of the disk which rotates with uniform angular speed in a clockwise direction. B is the point at which the tracing pen is set down in any initial phase of its rectilinear simple harmonic motion which has the same period as the disk, so that, if the disk did not rotate, the pen would move over the line ERG, its distance from R, its middle point, being at any moment the sine of the phase.

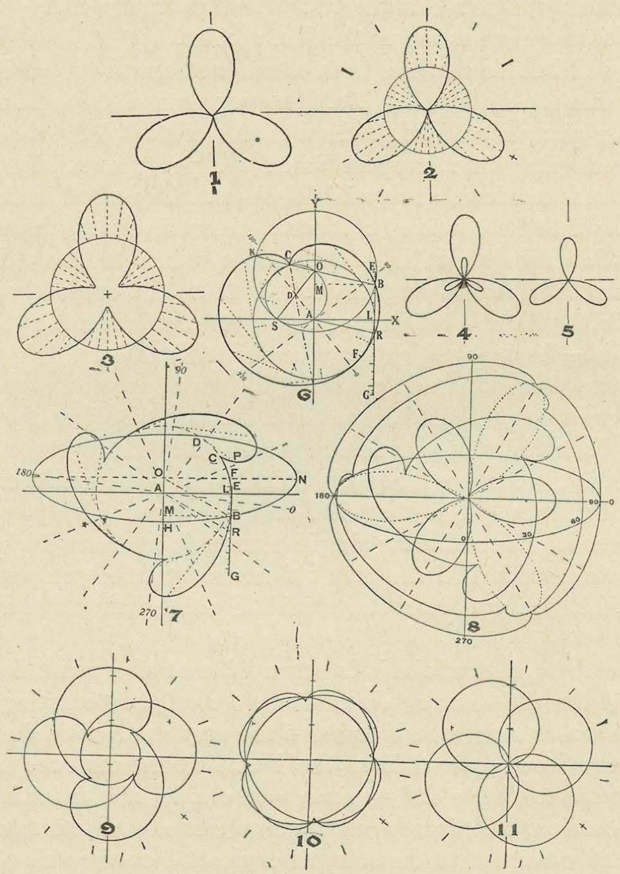
In order to draw a cardioid it is essential that the point B, at which the pen is set down on the rotating disk, should be on the "starting circle" whose radius OB is taken as unity and is equal to the amplitude of the harmonic motion ER or RG, and whose center O is on the Y axis at the distance from the center of the disk equal to $OA = BR = \sin \alpha =$ the sine of the initial phase. The cusp C of the cardioid will then be on the cusp circle, with radius one half of OB, which is internally tangent to the starting circle at S on the axis of X so that the angle $OSA = \alpha$, and which passes through O and A. The selection of the point B on the starting circle fixes the starting angle $\beta = AOB$, which then locates the cusp C on the diameter through B. The distance of the cusp C from A is equal to $AC = AL = MB = \sin \beta$. There is also a second starting point K, diametrically opposite B, so that $\beta' = 180° + \beta$, upon which the pen may be set down at the same initial phase α and trace the same cardioid.

The mechanical method mentioned of tracing the cardioid is what we might call the *tangent* method.[1] A similar method will be seen to apply in a more gen-

[1] Cf. "Concerning a new Method, etc."

eralized form to cuspidal rosettes of all kinds. To facilitate the comparison, the same letters and symbols will be used as much as possible.

Cuspidal Rosettes: the Ratio $n = 3$.—We begin with a simple ratio of $n = 3$, or $n = p/q = 3/1$, that is, p or 3 cycles of the rectilinear motion of the pen being equal to q or one of the disk, Fig. 7. It is immaterial whether we suppose both the pen and the disk to move in the manner indicated, or imagine the disk at rest and credit the pen with its motion in the reverse direction. The initial phase α of the rectilinear motion in Fig. 7 has been taken as 20°, so that



$BR = OA = \sin 20°$. We will take $\beta = 30°$. In comparing Figs. 6 and 7 we may note generic similarities and obvious differences. The starting point B and the rectilinear path ERG are the same in both figures, although the scale is not. The starting circle with radius unity in Fig. 6 becomes the large ellipse in Fig. 7, whose center O is as before on the Y axis at the distance $\sin \alpha$ from A. The conjugate axis of the ellipse OH is unity, and its transverse axis $ON = n = 3$, and is parallel to the X axis. AL is now $n \sin \beta = 3 \sin \beta$.

The Starting Point B.—That the point B must be on the ellipse mentioned may be proved by substituting the general ratio n for its value of unity in the cardioid. If in Fig. 6 we draw BO parallel to RA, these lines are equal, and $OA = BR = \sin\alpha$. Drawing BM parallel to AL makes them also equal to one another, and OM equal to LR. As LR is the sine of some phase of the harmonic motion, we may take it as the cosine of the angle ARL or its equal MOB, that is, as $\cos\beta$. Then $MB = AL = \sin\beta$ and $OB = AR = OS =$ unity. Therefore the starting point B is on the circle with radius unity and with its center O the distance $\sin\alpha$ from A.

In the case of a rosette as in Fig. 7, we have MB also equal to AL, $OA = BR = \sin\alpha$ and $LR = OM$. Taking these last equal to $\cos\beta$ as before, we cannot now make AL and MB equal to $\sin\beta$, but must take them equal to $n\sin\beta$, because the variation of the phase of the pen along the line ERG is now n times the angular speed of the disk. Hence the starting point B is on the ellipse whose semi-major axis $ON = n$ and semi-minor axis $OH =$ unity, and center O at distance $\sin\alpha$ from A.

The Locus of the Cusps C.—The rosette in Fig. 7 has three cusps, and in general it is evident that the number of cusps must be equal to p, the number of cycles of the pen, and that their angular intervals must be $360°/p$. For this reason we may confine ourselves to the cusp C that is first drawn after the pen has been set down at B and call it *the* cusp.

To find the phase of the pen when it is at the cusp, we observe that the pen must then be momentarily at rest, so that its rectilinear speed must be equal and opposite to its rotary velocity. As the latter is always at right angles to the radius through A, the harmonic motion can be so only when the pen crosses the X axis at L, so that one value of RL is the sine of the phase of the pen when at the cusp C and the other when it is at F, where the linear and rotary speeds are also equal but in the same direction. The rotary speed of the pen when at C is $ACd\theta = ALd\theta = n\sin\beta d\theta = \sin\beta d\beta$, its rectilinear speed at L is $d(LR)$ so that $LR = \int\sin\beta d\beta$ is equal to $\cos\beta$ or $\sin(90° \pm \beta)$ in absolute value. Of these two, $90° + \beta$ must be the phase for the cusp C, because as the rotary motion carries the pen anticlockwise, the rectilinear motion at L in Fig. 7 must then carry it in the opposite direction, that is, clockwise or downward, so that the phase must be greater than 90°. For the point F the phase must then be $90° - \beta$.

The phase of the harmonic motion at the cusp C being $90° + \beta$, that of the rotary motion must be $(90° + \beta)/n$. The angular position of C on the disk therefore varies directly as β, and as its linear distance from A is $n\sin\beta$, its locus must have an equation like $\rho = n\sin n\theta$, and when $n = 3$ as in Figs. 7 and 8, this is the equifoliated and non-cuspidal trifolium shown in dotted lines in Fig. 8, which is exactly like Fig. 5 but n times as large. The position of the axis of the first lobe is found by making $n\sin\beta$ a maximum, that is, $\beta = 90°$, so that (see Fig. 8 in which four rosettes are given with $\beta = 0°, 30°, 60°, 90°$) the phase of the pen $90° + \beta$ becomes $90° + 90° = 180°$, and the phase of the disk $180°/n$.

This last value directly locates the 0° phase of the disk, which may however be found also from any value of β, since it is equal to $(90° + \beta)/n$ as said before.

The arc AC of the cusp folium is equal to the elliptic arc HB on Fig. 7, and similarly to any corresponding elliptic arc on Fig. 8, that is to say, the distance of the cusp C from the center of the disk A as measured along the arc of the cusp folium is equal to that of the starting point B from the Y axis as measured along the ellipse. In the rectification of a polar curve the length of an arc is

$$s = \int (d\rho^2 + \rho^2 d\theta^2)^{1/2},$$

so that in the cusp folium where $\rho = n \sin n\theta$, we have

$$s = \int (n^2 \cos^2 n\theta \cdot n^2 d\theta^2 + n^2 \sin^2 n\theta \cdot d\theta^2)^{1/2}$$

$$= n \int_0^\beta \left(1 - \frac{n^2 - 1}{n^2} \sin^2 n\theta\right)^{1/2} n d\theta,$$

which is like

$$s = a \int_{\phi_0}^{\phi_1} (1 - e^2 \sin^2 \varphi)^{1/2} d\varphi,$$

the length of an arc of an ellipse with semi-major axis a, eccentricity e, and eccentric angle φ, as given by Byerly in his *Integral Calculus*, page 121, because in our ellipse $a = n$, $b = 1$, $e^2 = (a^2 - b^2)/a^2 = (n^2 - 1)/n^2$. Therefore the cusp folium arc AC equals the elliptic arc HB, and the perimeter of one lobe of the cusp folium is equal in length to the semiperimeter of the ellipse. This expression is true for values of $n > 1$, but when $n < 1$ it takes the form

$$s = \int (1 - (1 - n^2) \cos^2 n\theta)^{1/2} n d\theta$$

$$= b \int (1 - e^2 \cos^2 \varphi)^{1/2} d\varphi,$$

so that as a and b then exchange names and a cosine appears in the place of a sine, the elliptic arc is now reckoned from the end of the major instead of the minor axis, the zero point H however being always on the axis of Y.

The Locus of the Points of Contact F.—As the phase of the pen is $90° + \beta$ at the cusp C and $90° - \beta$ at the point of contact F, all that has been said about the former applies to the latter, provided we reverse the sign of β. The F points are thus seen to lie on the lobes of an equal equifoliated rosette with the equation $\rho = - n \sin n\theta$, these lobes being diametrically opposite to those of the cusp rosette. It has been omitted from Fig. 8.

The Tangent Method of Tracing the Rosette.—The mechanical method of tracing a rosette by having a pen move with a rectilinear simple harmonic motion of n cycles over a uniformly rotating disk, is mathematically a tangent method. If we imagine the two components of the pen's motion, the rectilinear over the line *ERG* and the rotary about *A*, to act successively instead of simultaneously, the pen is first at *L* in Fig. 7 in its rectilinear motion, in phases $90° + \beta$ and $90° - \beta$, and is then carried to *C* and *F* by the rotary motion on a circle with center *A* and radius $n \sin \beta$. The length of the tangent to this circle is then zero. At any other phase θ the pen in its harmonic motion is, say, at B', which may be anywhere, but which we may place on *B* in order not to congest the

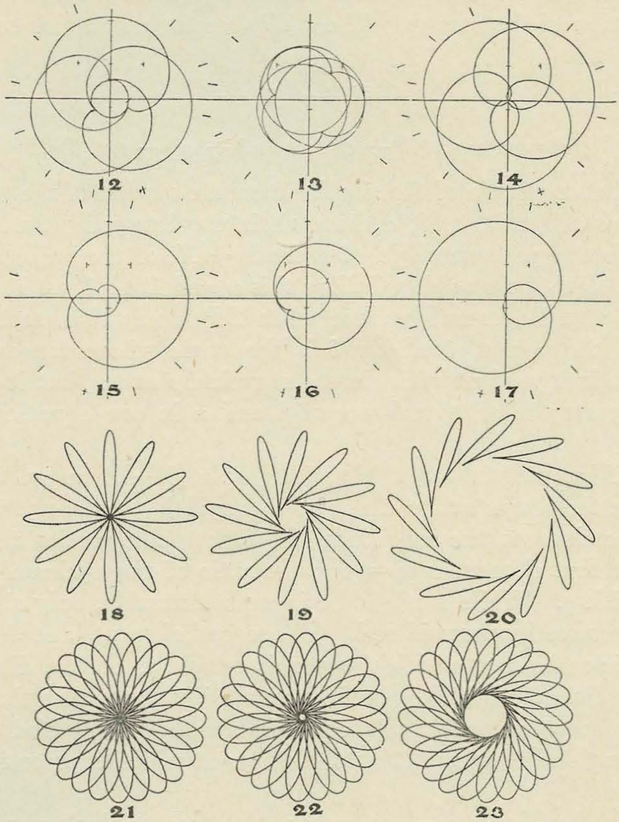


figure and which we accent in order to distinguish the two. While the rotary motion alone may be conceived first to carry the pen to *D*, the tangential harmonic motion then moves it along the tangent *DP* which is equal to $LB' = LR - B'R = \cos \beta - \sin \theta$, or rather $\sin \theta - \cos \beta$, because the tangent *DP* in the case illustrated is really negative.

On Fig. 7 these tangents have been drawn to almost every 30° of the disk. Owing to the small numerical value of *LE* in the instance presented, the tangents to the *tangent* circle, as we may call it, are positive only between *F* and *C*, between phases $90° - \beta$ and $90° + \beta$, which are here 60° and 120°, and negative for all other phases. The points *C* and *F* on the curve in phases $90° \pm \beta$ are thus at minimum, and those in phases 90° and 270° at maximum distances from A. It is obvious, of course, that as the harmonic motion is n times as rapid as the circular, the tangent at any phase angle on the tangent circle runs to a point on the curve in n times that angle, so that 60° on the circle in Fig. 7 is joined to $3 \times 60° = 180°$ on the rosette, 330° on the circle to $3 \times 330° = 2 \times 360° + 270°$ on the curve, and so on.

The Equation of a Cuspidal Rosette.—The tangent DP in Fig. 7 ought to lead us to the equation of the rosette. The radius vector ρ or AP is seen to be such that

$$\overline{AP}^2 = \overline{AD}^2 + \overline{DP}^2$$

or

$$\rho^2 = a^2n^2 \sin^2 \beta + a^2 (\sin (n\theta + \alpha) - \cos \beta)^2$$

in which a is the amplitude of the simple harmonic motion of the pen ER, which we may take as our unit, n, β, α, have their usual meaning and any assumed values, and θ is the position angle of D from $+ X$, or the angle DAL. But it is the angle PAL that we need. Calling this ω and DAP δ, we have $\omega = \theta + \delta$, using the plus sign in general because when DP is negative, as it is in Fig. 7, it will reverse the sign of δ. Now δ is the angle whose tangent is DP/AD. Taking θ as the independent variable, we may find ρ and ω, but it does not seem possible to express the relation between ρ and ω in one equation without the help of θ.

The Three Elements of a Cuspidal Rosette, n, α, β.—There are three elements that determine the shape and position of a cuspidal rosette, not to mention its size which depends upon the amplitude of the harmonic motion that we take as our unit. The first element is $n = p/q$, the ratio of the cycles of the pen p to those of the disk q. While n might be incommensurable, only simple ratios of integral numbers are here considered. The second element is α, the initial phase in its harmonic motion at which the pen is set down on the disk. This may have any value from 0° to 360°. In this paper α is taken as less than 90°, greater values having been treated in the cardioid article. The third element is β, the eccentric angle of the point B on the ellipse at which the pen is set down on the disk.

Variation in the Elements.—The nature of a rosette obviously depends upon $n = p/q$ and β, since p determines the number of lobes and cusps and q the number of its convolutions about the center A, while β, that is, $n \sin \beta$, modifies its shape by determining its distance from the center. There is then nothing left for α to do but to fix the position of the curve. For if the pen is first started in phase 0°, and then in phase α, the advance of the pen on the disk will be α/n, and the whole rosette will be shifted that angular amount forward in a clockwise direction. Hence the disk reading for $+ X$ then becomes α/n instead of 0°, as it is in Fig. 8, where $\alpha = 0°$. In Fig. 7 therefore, where $\alpha = 20°$ and $n = 3$, the circle reading for $+ X$ is $\alpha/n = 6° 40'$. The four points of the circle, 0°, 90°, 180°, 270° may thus be properly marked and intermediate radii drawn for every 30° at pleasure. Then, as the circle reading for the axis of the first cusp lobe was found to be $180°/n$, its position angle from $+ X$ then becomes $(180° - \alpha)/n$.

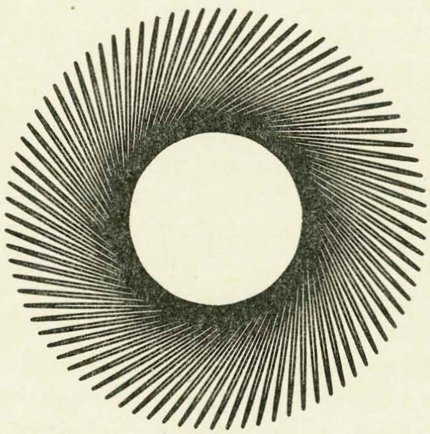
A variation in α alone shifts the center O of the ellipse along the Y axis to the distance $\sin \alpha$. It has no effect on the nature of the curve, as has been said, so that the rosette in Fig. 7, in which $n = 3$ and $\beta = 30°$ but $\alpha = 20°$, is exactly equal in every respect except as to the axes of X and Y to the second rosette on Fig. 8 in which also $n = 3$ and $\beta = 30°$ but $\alpha = 0°$. Even the position is the same in regard to the circle reading, because this is $(90° + \beta)/n$ for the cusp, independent of α, as we saw before.

A change of α, as has been said, affects merely the position of the rosette, so that it is drawn sooner or later than it was at the first value of α. While the curve is being drawn there is then no reason why we should not be able to take any instantaneous position of the pen as an initial position. As this initial position must be at the assumed starting point B at the eccentric angle β of the starting ellipse mentioned before whose center is the distance $\sin \alpha$ from A in the direction of B from R (Fig. 7), we may take this assumed point B on the ellipse, move it together with the ellipse along the rosette to a point in another phase, swing the ellipse about this second position of B until its center is at the distance of the sine of this new phase from A on a line parallel to the direction of the harmonic motion of the pen at the moment, and then mark the center on the paper. It we do this for all points of the rosette, we shall find the locus of the ellipse center O to be an equifoliated non-cuspidal rosette exactly like Fig. 5 when $n = 3$, with its lobe axes of unit magnitude lying on those of the contact F folium. The reason is that $AO = \sin \alpha$, and the angular revolution of the ellipse about B must be uniform for equal phase intervals, so that $d\theta$ is constant when $d\alpha$ is, as we always consider it to be. This O-folium may readily be drawn for any value of n, if we place the pen at A when in phase 0°. No illustrations of the O-folia are here given, since they would be for all values of n, as much like the original rosette when $\beta = 0°$ or 180° as Fig. 5 is like Fig. 1, that is to say, the O-folia would be non-cuspidal and of unit magnitude.

A variation in β alone does not affect the nature of the rosette when the old value of β is added to or subtracted from 180° and 360°. For this reason it is most convenient to use values of β less than 90°, and to express greater values in the way indicated. For $180° + \beta$ the new curve is symmetrical to the β rosette with respect to the center of the disk A. For $180° - \beta$ and $360° - \beta$ it is symmetrical to the β curve with regard to the $0° - 180°$ and $90° - 270°$ diameters respectively of the circle reading.

But it is the variations in n that teach us most about rosettes. Accordingly Figs. 9–25 show some typical cases. For Figs. 9–17 the initial phase α has been taken as 52° for the sake of comparison. The starting point B and the center O of the starting ellipse and the extremities of its axes have everywhere been marked. The dozen radiating dashes indicate every 30° of the circle, the 0° being marked by a cross. This last applies also to Fig. 2, in which however $\alpha = 90°$ and $\beta = 0°$. In Figs. 9, 10, 11, $n = 4/3$ and $\beta = 30°, 90°, 180°$, while in Figs. 12, 13, 14, $n = 3/4$ with the same values of β. In Figs. 15, 16, 17, $n = 1/2$ and $\beta = 30°, 60°, 180°$. When the denominator q in n is large, the number of convolutions of the curve are uninterestingly numerous in proportion.

As α determines only the position of a curve, and this is generally of no consequence, the most convenient value of α to use is 90° when $\sin \alpha$ is a maximum and the pen is at one extremity of its harmonic path. Then when n is large and β small, the starting point is practically on the X axis. Thus Figs. 18, 19, 20 show the ratio $n = 12$ and $\beta = 0°, 3°, 19°$, respectively. In Figs. 21, 22, 23, $n = 24/5$ and $\beta = 0°, 1° 4', 5° 42'$, and in Figs. 24 and 25 $n = 84$, $\beta = 0° 38'$ and $\pm 0° 38'$. This ratio of $n = 84$ is the largest the author's machine is at present able to produce. The machine was described in the *Scientific American Supplement* of February 9 and 16, 1918. Fig. 25 is in a sense merely the double



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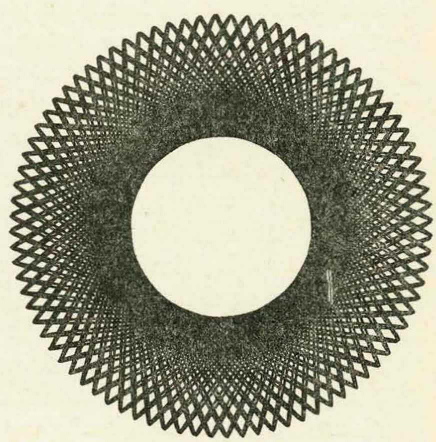

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of Fig. 24. It was drawn by first tracing Fig. 24 with the B point to the right of the center, and then by starting the pen an equal distance to the left. The cuspidal nature of some of the rosettes presented is not conspicuous, especially when β is very small, on account of the closeness and apparent superpositions of the paths of the pen when near the cusp. While other and more beautiful rosettes could have been drawn, had the restriction as to cusps been waived, they would have been foreign to the present study.